Einstein–Rosen "bridge" needs lightlike brane source

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ABSTRACT

The Einstein–Rosen "bridge" wormhole solution proposed in the classic paper (Einstein and Rosen (1935) [1]) *does not* satisfy the vacuum Einstein equations at the wormhole throat. We show that the fully consistent formulation of the original Einstein–Rosen "bridge" requires solving Einstein equations of bulk $D = 4$ gravity coupled to a lightlike brane with a well-defined world-volume action. The non-vanishing contribution of Einstein–Rosen "bridge" solution to the right-hand side of Einstein equations at the throat matches precisely the surface stress–energy tensor of the lightlike brane which automatically occupies the throat ("horizon straddling") — a feature triggered by the world-volume lightlike brane dynamics.

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1. Introduction

The Einstein–Rosen "bridge" space–time proposed in the classical paper [1] is historically one of the first examples of what later became known as wormhole space–time manifolds (for a review, see [2,3] and references therein).

In a series of recent papers [4,5] we have explored the novel possibility of employing *lightlike branes* (*LL-branes* for short) as natural self-consistent gravitational sources for traversable wormhole space–times, in other words, generating wormhole solutions in self-consistent bulk gravity-matter systems coupled to *LL-branes* through dynamically derived world-volume *LL-brane* stress–energy tensors. Namely, we have provided in [4,5] a systematic general scheme to construct self-consistent spherically symmetric or rotating cylindrical wormhole solutions via *LL-branes*, such that the latter occupy the wormhole throats and match together two copies of exterior regions of spherically symmetric or rotating cylindrical black holes (the regions beyond the outer horizons). These wormhole solutions combine the features of the original Einstein–Rosen "bridge" manifold [1] (wormhole throat located at horizon) with

the feature "charge without charge" of Misner–Wheeler wormholes [6]. They have been also shown to be *traversable* w.r.t. the *proper time* of travelling observers [5].

There exist several other types of physically interesting wormhole solutions in the literature generated by different types of matter and without horizons. For a recent discussion, see Ref. [7] and references therein.

As a particular case of our construction in [4,5], the matching of two exterior regions of Schwarzschild space–time at the horizon surface $r = 2m$ through an *LL-brane* turns out to be the self-consistent realization of the original Einstein–Rosen "bridge". Namely, the Einstein–Rosen "bridge" metric in its original form from [1] is *not* a solution of the vacuum Einstein equations but rather it requires the presence of an *LL-brane* source at $r = 2m$ — a feature not recognized in the original Einstein–Rosen work [1]. It is the main purpose of the present note to explain the latter in more detail.

Let us particularly emphasize that here and in what follows we consider the Einstein–Rosen "bridge" in its original formulation in Ref. [1] as a four-dimensional space–time manifold consisting of two copies of the exterior Schwarzschild space–time region matched along the horizon. On the other hand, the nomenclature of "Einstein–Rosen bridge" in several standard textbooks (*e.g.* [8]) uses the Kruskal–Szekeres manifold and it is *not equivalent* to the original construction in [1]. Namely, the two regions in Kruskal–Szekeres space–time corresponding to the outer Schwarzschild

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space–time region ($r > 2m$) and labeled (*I*) and (*III*) in [8] are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0$, $V = 0$ in Kruskal–Szekeres coordinates), whereas in the original Einstein–Rosen "bridge" construction the boundary between the two identical copies of the outer Schwarzschild space–time region ($r > 2m$) is a three-dimensional hypersurface ($r = 2m$).

In what follows we will make an essential use of the explicit world-volume Lagrangian formalism for *LL-branes* proposed earlier in Ref. [9]. There are several characteristic features of *LL-branes* which drastically distinguish them from ordinary Nambu–Goto branes:

(i) They describe intrinsically lightlike modes, whereas Nambu–Goto branes describe massive ones.
(ii) The tension of the *LL-brane* arises as an *additional dynamical degree of freedom*, whereas Nambu–Goto brane tension is a given *ad hoc* constant. This is an important feature significantly distinguishing our *LL-brane* models from the previously proposed *tensionless p*-branes (for a review, see Ref. [10]) which rather resemble a *p*-dimensional continuous distribution of massless point-particles.
(iii) Consistency of *LL-brane* dynamics in a spherically or axially symmetric gravitational background of codimension one requires the presence of an event horizon which is automatically occupied by the *LL-brane* ("horizon straddling" according to the terminology of Ref. [11]).
(iv) When the *LL-brane* moves as a *test* brane in spherically or axially symmetric gravitational backgrounds its dynamical tension exhibits exponential "inflation/deflation" time behaviour [12] − an effect similar to the "mass inflation" effect around black hole horizons [13].

Let us also note that *LL-branes* by themselves play an important role in the description of various physically important phenomena in general relativity, such as impulsive lightlike signals arising in cataclysmic astrophysical events [14], the "membrane paradigm" [15] of black hole physics and the thin-wall approach to domain walls coupled to gravity [11,16,17] (see also [18]).

In Section 2 below we show that the original Einstein–Rosen "bridge" metric fails to satisfy the vacuum Einstein equations due to the appearance on the r.h.s. of a non-vanishing ill-defined (as distribution) δ-function singularity at the throat. This indicates the presence of some kind of matter source concentrated on the throat − a three-dimensional lightlike hypersurface connecting the two "universes" carrying the geometry of the exterior Schwarzschild space–time region.

In Section 3 we briefly review the main properties of the *LL-brane* world-volume Lagrangian dynamics which are of utmost importance for our main construction in the following Section 4. We propose there a different metric describing the Einstein–Rosen "bridge" manifold, which satisfy the Einstein equations with a well-defined δ-function contribution on the r.h.s. identified as the stress–energy tensor of an *LL-brane* coupled to bulk gravity. As mentioned above, the latter construction is a particular case of the general construction of spherically and rotating cylindrical wormholes via *LL-branes* proposed in [4,5].

In Section 5 we provide an alternative construction of Einstein–Rosen "bridge" space–time as a *lightlike* limit of spherically symmetric wormhole with a *timelike* "thin shell" at the throat. In particular, we explain the reason why we are not encountering any divergencies in the lightlike limit when joining two exterior Schwarzschild regions along timelike "thin shell" with our choice of coordinates unlike the case with the standard treatment using Gaussian normal coordinates (see *e.g.* [2]; the latter formalism was specifically designed for timelike "thin shells", without having in

mind a lightlike limit). For an alternative approach appropriate also for lightlike "thin shells", see Refs. [11,19]. The principal difference w.r.t. the present formalism (Ref. [9] and Section 3 below) is that in our case the *LL-brane* dynamics is systematically derived from world-volume action principle.

2. Einstein–Rosen "bridge" fails to satisfy vacuum Einstein equations

Let us start with the coordinate system proposed in [1], which is obtained from the original Schwarzschild coordinates by defining $u^2 = r - 2m$, so that the Schwarzschild metric becomes:

$$ds^2 = -\frac{u^2}{u^2 + 2m}(dt)^2 + 4(u^2 + 2m)(du)^2 + (u^2 + 2m)^2((d\theta)^2 + \sin^2\theta(d\varphi)^2). \tag{1}$$

Then Einstein and Rosen take two identical copies of the exterior Schwarzschild space–time region ($r > 2m$) by letting the new coordinate u to vary between $-\infty$ and $+\infty$ (*i.e.*, we have the same $r \geqslant 2m$ for $\pm u$). The two Schwarzschild exterior space–time regions must be matched at the horizon $u = 0$.

Let us examine whether the original Einstein–Rosen solution satisfy the vacuum Einstein equations everywhere. To this end let us consider the Levi-Civita identity (see *e.g.* [20]):

$$R^0_0 = -\frac{1}{\sqrt{-g_{00}}}\nabla^2\left(\sqrt{-g_{00}}\right) \tag{2}$$

valid for any metric of the form $ds^2 = g_{00}(r)(dt)^2 + h_{ij}(r, \theta, \varphi) \times dx^i \, dx^j$ and where ∇^2 is the three-dimensional Laplace–Beltrami operator $\nabla^2 = \frac{1}{\sqrt{h}}\frac{\partial}{\partial x^i}(\sqrt{h}h^{ij}\frac{\partial}{\partial x^j})$. The Einstein–Rosen metric (1) solves $R^0_0 = 0$ for all $u \neq 0$. However, since $\sqrt{-g_{00}} \sim |u|$ as $u \to 0$ and since $\frac{\partial^2}{\partial u^2}|u| = 2\delta(u)$, Eq. (2) tells us that:

$$R^0_0 \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2), \tag{3}$$

and similarly for the scalar curvature $R \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2)$. From (3) we conclude that:

(i) The non-vanishing r.h.s. of (3) exhibits the explicit presence of some lightlike matter source on the throat − an observation which is missing in the original formulation [1] of the Einstein–Rosen "bridge". In fact, the problem with the metric (1) satisfying the vacuum Einstein equations at $u = 0$ has been noticed in Ref. [1], where in Eq. (3a) the authors multiply Ricci tensor by an appropriate power of the determinant g of the metric (1) vanishing at $u = 0$ so as to enforce fulfillment of the vacuum Einstein equations everywhere, including at $u = 0$.
(ii) The coordinate u in (1) is *inadequate* for description of the original Einstein–Rosen "bridge" at the throat due to the *ill-definiteness* as distribution of the r.h.s. in (3).

We will now describe an alternative construction of the Einstein–Rosen "bridge" wormhole as a spherically symmetric wormhole with Schwarzschild geometry produced via *LL-brane* sitting at its throat in a self-consistent formulation, namely, solving Einstein equations with a surface stress–energy tensor of the lightlike brane derived from a well-defined world-volume brane action. Moreover, we will show that the mass parameter m of the Einstein–Rosen "bridge" is not a free parameter but rather is a function of the *dynamical LL-brane* tension.

To this end we will employ the Finkelstein–Eddington coordinates for the Schwarzschild metric [21] (see also [8]):

$$ds^2 = -A(r)(dv)^2 + 2\,dv\,dr + r^2\big[(d\theta)^2 + \sin^2\theta(d\varphi)^2\big];$$

$$A(r) = 1 - \frac{2m}{r}. \tag{4}$$

The advantage of the metric (4) over the metric in standard Schwarzschild coordinates is that both (4) as well as the corresponding Christoffel coefficients *do not* exhibit coordinate singularities on the horizon ($r = 2m$).

Let us introduce the following modification of (4):

$$ds^2 = -\tilde{A}(\eta)(dv)^2 + 2\,dv\,d\eta + \tilde{r}^2(\eta)\big[(d\theta)^2 + \sin^2\theta(d\varphi)^2\big], \tag{5}$$

where we substituted r with a new coordinate η via $r = 2m + |\eta|$, *i.e.*:

$$\tilde{A}(\eta) = A(2m + |\eta|) = \frac{|\eta|}{|\eta| + 2m}, \qquad \tilde{r}(\eta) = 2m + |\eta|. \tag{6}$$

The metric describes two identical copies of Schwarzschild *exterior* space–time region ($r > 2m$), which correspond to $\eta > 0$ and $\eta < 0$, respectively, and which are "glued" together at the horizon $\eta = 0$ (*i.e.*, $r = 2m$), where the latter will serve as a throat of the overall wormhole solution. This is precisely the space–time manifold of the original Einstein–Rosen "bridge" construction (*cf.* Eq. (1)) in terms of the Eddington–Finkelstein coordinate system.

As we will see in what follows, the appropriate coordinate to describe the full Einstein–Rosen "bridge" manifold (including at the wormhole throat) is precisely η rather than the original Einstein–Rosen's coordinate u. Obviously, the metric (5)–(6) is smooth everywhere except at the horizon $\eta = 0$ where it is only continuous but not differentiable. Therefore, it is clear that the pertinent Ricci tensor and the scalar curvature will exhibit well-defined distributional contributions $\sim \delta(\eta)$ due to the terms containing second order derivatives w.r.t. η (because of $\partial_\eta^2|\eta| = 2\delta(\eta)$), in other words, there must be some *lightlike* "thin shell" matter present on the horizon.

3. World-volume Lagrangian formulation of lightlike branes

In a series of previous papers [4,9,12] we proposed and studied manifestly reparametrization invariant world-volume actions describing intrinsically lightlike p-branes for any world-volume dimension $(p + 1)$:

$$S = -\int d^{p+1}\sigma\,\Phi\left[\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - L(F^2)\right]. \tag{7}$$

Here the following notions and notations are used:

- Alternative non-Riemannian integration measure density Φ (volume form) on the p-brane world-volume manifold:

$$\Phi \equiv \frac{1}{(p+1)!}\varepsilon^{a_1\ldots a_{p+1}} H_{a_1\ldots a_{p+1}}(B),$$

$$H_{a_1\ldots a_{p+1}}(B) = (p+1)\partial_{[a_1} B_{a_2\ldots a_{p+1}]} \tag{8}$$

  instead of the usual $\sqrt{-\gamma}$. Here $\varepsilon^{a_1\ldots a_{p+1}}$ is the alternating symbol ($\varepsilon^{01\ldots p} = 1$), γ_{ab} ($a,b = 0,1,\ldots,p$) indicates the intrinsic Riemannian metric on the world-volume, and $\gamma = \det\|\gamma_{ab}\|$. $H_{a_1\ldots a_{p+1}}(B)$ denotes the field-strength of an auxiliary world-volume antisymmetric tensor gauge field $B_{a_1\ldots a_p}$ of rank p. Note that γ_{ab} is *independent* of the auxiliary world-volume fields $B_{a_1\ldots a_p}$. The alternative non-Riemannian volume form (8) has been first introduced in the context of modified standard (non-lightlike) string and p-brane models in Ref. [22].

- $X^\mu(\sigma)$ are the p-brane embedding coordinates in the bulk D-dimensional space–time with bulk Riemannian metric $G_{\mu\nu}(X)$ with $\mu,\nu = 0,1,\ldots,D-1$; $(\sigma) \equiv (\sigma^0 \equiv \tau, \sigma^i)$ with $i = 1,\ldots,p$; $\partial_a \equiv \frac{\partial}{\partial\sigma^a}$.

- g_{ab} is the induced metric:

$$g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X), \tag{9}$$

  which becomes *singular* on-shell (manifestation of the lightlike nature, *cf.* Eq. (14) below).

- $A_{a_1\ldots a_{p-1}}$ is an Auxiliary $(p-1)$-rank antisymmetric tensor gauge field on the world-volume with p-rank field-strength and its dual:

$$F_{a_1\ldots a_p} = p\,\partial_{[a_1} A_{a_2\ldots a_p]}, \qquad F^{*a} = \frac{1}{p!}\frac{\varepsilon^{aa_1\ldots a_p}}{\sqrt{-\gamma}} F_{a_1\ldots a_p}. \tag{10}$$

  Its Lagrangian $L(F^2)$ is *arbitrary* function of F^2 with the short-hand notation:

$$F^2 \equiv F_{a_1\ldots a_p} F_{b_1\ldots b_p} \gamma^{a_1 b_1}\cdots\gamma^{a_p b_p}. \tag{11}$$

Rewriting the action (7) in the following equivalent form:

$$S = -\int d^{p+1}\sigma\,\chi\sqrt{-\gamma}\left[\frac{1}{2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) - L(F^2)\right],$$

$$\chi \equiv \frac{\Phi}{\sqrt{-\gamma}} \tag{12}$$

with Φ the same as in (8), we find that the composite field χ plays the role of a *dynamical* (*variable*) *brane tension*.

Now let us consider the equations of motion corresponding to (7) w.r.t. $B_{a_1\ldots a_p}$:

$$\partial_a\left[\frac{1}{2}\gamma^{cd} g_{cd} - L(F^2)\right] = 0 \quad \longrightarrow \quad \frac{1}{2}\gamma^{cd} g_{cd} - L(F^2) = M, \tag{13}$$

where M is an arbitrary integration constant. The equations of motion w.r.t. γ^{ab} read:

$$\frac{1}{2}g_{ab} - F^2 L'(F^2)\left[\gamma_{ab} - \frac{F_a^* F_b^*}{F^{*2}}\right] = 0, \tag{14}$$

where F^{*a} is the dual field strength (10).

There are two important consequences of Eqs. (13)–(14). Taking the trace in (14) and comparing with (13) implies the following crucial relation for the Lagrangian function $L(F^2)$:

$$L(F^2) - pF^2 L'(F^2) + M = 0, \tag{15}$$

which determines F^2 (11) on-shell as certain function of the integration constant M (13), *i.e.*

$$F^2 = F^2(M) = \text{const.} \tag{16}$$

The second and most profound consequence of Eqs. (14) is that the induced metric (9) on the world-volume of the p-brane model (7) is *singular* on-shell (as opposed to the induced metric in the case of ordinary Nambu–Goto branes):

$$g_{ab} F^{*b} = 0, \tag{17}$$

i.e., the tangent vector to the world-volume $F^{*a}\partial_a X^\mu$ is *lightlike* w.r.t. metric of the embedding space–time. Thus, we arrive at the following important conclusion: every point on the surface of the p-brane (7) moves with the speed of light in a time-evolution along the vector-field F^{*a} which justifies the name *LL-brane* (lightlike brane) model for (7).

Before proceeding let us note that there exists a dynamically equivalent dual Nambu–Goto-type world-volume action [4,12] for the *LL-brane* producing the same equations of motion as the original Polyakov-type *LL-brane* action (7):

$$S_{\text{NG}} = -\int d^{p+1}\sigma\,T\sqrt{\left|\det\left\|g_{ab} - \epsilon\frac{1}{T^2}\partial_a u\,\partial_b u\right\|\right|}, \quad \epsilon = \pm 1, \tag{18}$$

where g_{ab} indicates the induced metric on the world-volume (9), u is the dual gauge potential w.r.t. $A_{a_1 \ldots a_{p-1}}$ ($F_a^*(A) = \text{const} \times \chi^{-1} \partial_a u$), and T is *dynamical* tension simply proportional to the dynamical tension in the Polyakov-type formulation (12).

World-volume reparametrization invariance allows to introduce the standard synchronous gauge-fixing conditions:

$$\gamma^{0i} = 0 \quad (i = 1, \ldots, p), \qquad \gamma^{00} = -1. \tag{19}$$

Also, we will use a natural ansatz for the "electric" part of the auxiliary world-volume gauge field-strength:

$$F^{*i} = 0 \quad (i = 1, \ldots, p), \quad i.e. \quad F_{0 i_1 \ldots i_{p-1}} = 0, \tag{20}$$

meaning that we choose the lightlike direction in Eq. (17) to coincide with the brane proper-time direction on the world-volume ($F^{*a} \partial_a \sim \partial_\tau$). The Bianchi identity ($\nabla_a F^{*a} = 0$) together with (19)–(20) and the definition for the dual field-strength in (10) imply:

$$\partial_0 \gamma^{(p)} = 0 \quad \text{where} \quad \gamma^{(p)} \equiv \det \|\gamma_{ij}\|. \tag{21}$$

Then *LL-brane* equations (14) acquire the form (recall definition of g_{ab} (9)):

$$g_{00} \equiv \dot{X}^\mu G_{\mu\nu} \dot{X}^\nu = 0, \qquad g_{0i} = 0, \qquad g_{ij} - 2a_0 \gamma_{ij} = 0 \tag{22}$$

(the latter are analogs of Virasoro constraints). Here a_0 is strictly positive M-dependent constant:

$$a_0 \equiv F^2 L'(F^2)\big|_{F^2 = F^2(M)} = \text{const} \tag{23}$$

($L'(F^2)$ denotes derivative of $L(F^2)$ w.r.t. the argument F^2). In particular, $a_0 = M$ for the "wrong-sign" Maxwell choice $L(F^2) = 1/4F^2$.

Consider now codimension one *LL-brane* moving in a general spherically symmetric background:

$$ds^2 = -A(t,r)(dt)^2 + B(t,r)(dr)^2 + C(t,r) h_{ij}(\vec{\theta}) d\theta^i d\theta^j, \tag{24}$$

i.e., $D = (p+1) + 1$, with the simplest non-trivial ansatz for the *LL-brane* embedding coordinates $X^\mu(\sigma)$:

$$t = \tau \equiv \sigma^0, \qquad r = r(\tau), \qquad \theta^i = \sigma^i \quad (i = 1, \ldots, p). \tag{25}$$

The *LL-brane* equations (22), taking into account (19)–(20), acquire the form:

$$-A + B\dot{r}^2 = 0, \quad i.e. \quad \dot{r} = \pm \sqrt{\frac{A}{B}}, \qquad \partial_t C + \dot{r} \partial_r C = 0. \tag{26}$$

In particular, we are interested in static spherically symmetric metrics in standard coordinates:

$$ds^2 = -A(r)(dt)^2 + A^{-1}(r)(dr)^2 + r^2 h_{ij}(\vec{\theta}) d\theta^i d\theta^j \tag{27}$$

for which Eqs. (26) yield:

$$\dot{r} = 0, \quad i.e. \quad r(\tau) = r_0 = \text{const}, \qquad A(r_0) = 0. \tag{28}$$

Eq. (28) tells us that consistency of *LL-brane* dynamics in a spherically symmetric gravitational background of codimension one requires the latter to possess a horizon (at some $r = r_0$), which is automatically occupied by the *LL-brane* ("horizon straddling").

Similar feature ("horizon straddling") occurs also for a codimension one *LL-brane* moving in axially or cylindrically symmetric (rotating) backgrounds [4,5].

4. Lightlike brane as a source of Einstein–Rosen "bridge" wormhole

We will now show that the newly proposed metric (5)–(6) is a self-consistent solution of Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 8\pi T_{\mu\nu}^{(\text{brane})} \tag{29}$$

derived from the action describing bulk ($D = 4$) gravity coupled to an *LL-brane*:

$$S = \int d^4x \sqrt{-G} \frac{R(G)}{16\pi} + S_{\text{LL}}, \tag{30}$$

where S_{LL} is the *LL-brane* world-volume action (12) with $p = 2$.

Using the simplest non-trivial ansatz for the *LL-brane* embedding coordinates $X^\mu \equiv (v, \eta, \theta, \varphi) = X^\mu(\sigma)$:

$$v = \tau \equiv \sigma^0, \qquad \eta = \eta(\tau), \qquad \theta^1 \equiv \theta = \sigma^1, \qquad \theta^2 \equiv \varphi = \sigma^2, \tag{31}$$

the pertinent *LL-brane* equations of motion yield:

$$\eta(\tau) = 0 \quad - \quad \text{horizon "straddling" by the } \textit{LL-brane}, \tag{32}$$

and the following expression for the *LL-brane* energy–momentum tensor:

$$T_{(\text{brane})}^{\mu\nu} = -\frac{2}{\sqrt{-G}} \frac{\delta S_{\text{LL}}}{\delta G_{\mu\nu}} = S^{\mu\nu} \delta(\eta),$$

$$S^{\mu\nu} = \frac{\chi}{2a_0} \left[\partial_\tau X^\mu \partial_\tau X^\nu - 2a_0 G^{ij} \partial_i X^\mu \partial_j X^\nu \right]_{v=\tau, \eta=0, \theta^i = \sigma^i}. \tag{33}$$

Here G^{ij} is the inverse metric in the ($\theta^i) \equiv (\theta, \varphi)$ subspace and a_0 indicates the integration constant parameter arising in the *LL-brane* world-volume dynamics (Eq. (23)).

Let us now turn to the Einstein equations (29) where we explicitly separate the terms contributing to δ-function singularities on the l.h.s., i.e., terms containing second-order derivatives w.r.t. η:

$$R_{\mu\nu} \equiv \partial_\eta \Gamma_{\mu\nu}^\eta - \partial_\mu \partial_\nu \ln \sqrt{-G} + \cdots = 8\pi \left(S_{\mu\nu} - \frac{1}{2} G_{\mu\nu} S_\lambda^\lambda \right) \delta(\eta) \tag{34}$$

(the dots indicating non-singular terms). Using the explicit expressions:

$$\Gamma_{vv}^\eta = \frac{1}{2} \tilde{A} \partial_\eta \tilde{A}, \qquad \Gamma_{v\eta}^\eta = -\frac{1}{2} \partial_\eta \tilde{A},$$

$$\Gamma_{ij}^\eta = -\frac{1}{2} \tilde{A} G_{ij} \partial_\eta \ln \tilde{r}^2, \qquad \sqrt{-G} = \tilde{r}^2 \tag{35}$$

with $\tilde{A}(\eta)$ and $\tilde{r}(\eta)$ as in (6), taking into account $\tilde{A}(0) = 0$ it is straightforward to check that non-zero δ-function contributions in $R_{\mu\nu}$ appear for $(\mu\nu) = (v\eta)$ and $(\mu\nu) = (\eta\eta)$ only. Substituting also the expressions for the components of the *LL-brane* stress–energy tensor (33) (with G_{ij} indicating the metric in the $(\theta^i) \equiv (\theta, \varphi)$ subspace):

$$S_{\eta\eta} = \frac{1}{2a_0} \chi, \qquad S_\lambda^\lambda = -2\chi, \qquad S_{ij} = -\chi G_{ij}, \tag{36}$$

the Einstein equations (34) yield for $(\mu\nu) = (v\eta)$ and $(\mu\nu) = (\eta\eta)$ the following matchings of the coefficients in front of the δ-functions, respectively:

$$m = \frac{1}{16\pi|\chi|}, \qquad m = \frac{a_0}{2\pi|\chi|}, \tag{37}$$

where the *LL-brane* dynamical tension must be *negative*. Consistency between the two relations (37) fixes the value $a_0 = 1/8$ for the integration constant a_0 (23). Most importantly, the first equation in (37) shows that the mass parameter of both Schwarzschild

"universes" is determined uniquely by the dynamical *LL-brane* tension.

From expressions (36) and the relation $S_\lambda^\lambda = 2\mathcal{P} - \rho$ the *LL-brane* pressure $\mathcal{P}$ and energy density ρ are identified to be:

$$S_{ij} = \mathcal{P}G_{ij} \;\rightarrow\; \mathcal{P} = |\chi|, \qquad \rho = 0. \tag{38}$$

At this point let us note that violation of the null energy condition takes place (the *LL-brane* being an "exotic matter") as predicted by general wormhole arguments (*cf.* Ref. [2]).

5. Einstein–Rosen "bridge" as a limit of spherically symmetric wormhole with a timelike shell at the throat

Let us now consider a different modification of Eddington–Finkelstein form of the Schwarzschild metric (4) (*cf.* (5)):

$$ds^2 = -\tilde{A}_1(\eta)(dv)^2 + 2\,dv\,d\eta + \tilde{r}_1^2(\eta)\big[(d\theta)^2 + \sin^2\theta\,(d\varphi)^2\big], \tag{39}$$

where:

$$\tilde{A}_1(\eta) = A\big(r_1 + |\eta|\big), \qquad \tilde{r}_1(\eta) = r_1 + |\eta|, \quad r_1 > 2m, \tag{40}$$

i.e., we now introduce a different change of coordinates from r to η via $r = r_1 + |\eta|$. The metric (39)–(40) describes two identical copies of Schwarzschild *exterior* space–time region ($r > r_1 \;(> 2m)$), which correspond to $\eta > 0$ and $\eta < 0$, respectively, and which are "glued" together at the timelike hypersurface $\eta = 0$ (*i.e.*, $r = r_1 > 2m$). The latter hypersurface will serve as a "throat". As above the metric (39) is smooth everywhere except at the "throat" $\eta = 0$ where it is only continuous but not differentiable. Therefore, once again the Ricci tensor and the scalar curvature will exhibit distributional contributions $\sim \delta(\eta)$ due to the terms containing second order derivatives w.r.t. η, in other words, they will indicate the presence of a *timelike* "thin shell" matter on the throat.

Using Eqs. (35) for the current metric (*i.e.*, replacing $\tilde{A}$ with $\tilde{A}_1$ from (40)) we find the following results for the pertinent Ricci tensor components:

$$R_{vv} = \frac{8m}{r_1^2}\left(1 - \frac{2m}{r_1}\right)\delta(\eta), \qquad R_{v\eta} = -\frac{2m}{r_1^2}\delta(\eta),$$

$$R_{\eta\eta} = -\frac{4}{r_1}\delta(\eta), \qquad R_{ij} = -\frac{2G_{ij}}{r_1}\left(1 - \frac{2m}{r_1}\right)\delta(\eta),$$

$$R_{vi} = 0, \qquad R_{\eta i} = 0. \tag{41}$$

Writing the Einstein equations for the metric (39)–(40) in the form:

$$R_{\mu\nu} = 8\pi\left(S_{\mu\nu} - \frac{1}{2}G_{\mu\nu}S_\lambda^\lambda\right)\delta(\eta) \tag{42}$$

and comparing with (41), we identify the following timelike "thin shell" matter stress–energy tensor $S_{\mu\nu}$:

$$S_{vv} = \frac{1}{2\pi r_1^2}\left(1 - \frac{2m}{r_1}\right)\left(\frac{7m}{2} - r_1\right), \qquad S_{v\eta} = \frac{1}{2\pi r_1}\left(1 - \frac{2m}{r_1}\right),$$

$$S_{\eta\eta} = -\frac{1}{2\pi r_1}, \qquad S_{ij} = \frac{G_{ij}}{4\pi r_1}\left(1 - \frac{m}{r_1}\right),$$

$$S_{vi} = 0, \qquad S_{\eta i} = 0. \tag{43}$$

In the limit $r_1 \to 2m$ when the timelike "thin shell" (the throat) is moved to the horizon, thus becoming a lightlike "thin shell", the only surviving non-vanishing components of $S_{\mu\nu}$ read:

$$S_{\eta\eta} = -\frac{1}{4\pi m}, \qquad S_{ij} = \frac{G_{ij}}{16\pi m}, \qquad S_\lambda^\lambda = \frac{1}{8\pi m}. \tag{44}$$

Comparing Eqs. (44) with Eqs. (36) and accounting for (37), we see that the lightlike limit of the "thin shell" matter at the throat coincides exactly with the *LL-brane* with dynamical tension whose

dynamics is consistently described by the world-volume action S_{LL} (12) appearing in (30). Yet, unlike the case with the *LL-brane*, the stress–energy tensor of the timelike "thin-shell" (before the lightlike limit) *is not* derived from any independent timelike world-volume brane Lagrangian.

Let us emphasize that we did not encounter any divergencies when taking the lightlike limit $r_1 \to 2m$ above unlike the case with the usual procedure for glueing together two outer Schwarzschild regions (with $r > r_1 \;(> 2m)$) along a timelike "thin shell" throat at $r = r_1$ (Ref. [2, Section 15.2.3]). The reason is that the Gaussian normal coordinate $\bar{\eta}$ used to describe the normal direction w.r.t. hypersurface of the timelike "thin shell" turns out to be *inappropriate* in the lightlike limit. Indeed, let us compare the above construction of Einstein–Rosen-like wormhole with a *timelike* throat given by the metric (39)–(40) (Eqs. (41)–(43)) against the standard construction using Gaussian normal coordinate $\bar{\eta}$ (*cf.* Eqs. (15.40)–(15.41) in Ref. [2]):

$$ds^2 = -\bar{A}(\bar{\eta})(dt)^2 + (d\bar{\eta})^2 + \bar{r}^2(\bar{\eta})\big[(d\theta)^2 + \sin^2\theta\,(d\varphi)^2\big], \tag{45}$$

where:

$$\bar{A}(\bar{\eta}) = 1 - \frac{2m}{\bar{r}(\bar{\eta})}, \qquad \frac{d\bar{r}}{d\bar{\eta}} = \sqrt{\bar{A}(\bar{\eta})}. \tag{46}$$

The transformation between the coordinates $x \equiv (v, \eta, \theta, \varphi)$ and $\bar{x} \equiv (t, \bar{\eta}, \theta, \varphi)$ relating the metrics (39) and (45) is:

$$t = v - r_1 - |\eta| - 2m\ln\left|\frac{r_1 + |\eta|}{2m} - 1\right|,$$

$$\bar{\eta} = \bar{\eta}(\eta) \quad \text{where} \quad \frac{d\bar{\eta}}{d\eta} = \frac{1}{\sqrt{\tilde{A}(\eta)}} \tag{47}$$

with $\tilde{A}$ as in (40), and θ, φ – unchanged. Accordingly, the $D = 4$ energy–momentum tensor of the "thin shell" transforms as $T_{\mu\nu}(x) = \bar{T}_{\kappa\lambda}(\bar{x})\frac{\partial \bar{x}^\kappa}{\partial x^\mu}\frac{\partial \bar{x}^\lambda}{\partial x^\nu}$ with $T_{\mu\nu}(x) = S_{\mu\nu}(v, \theta, \varphi)\delta(\eta)$ and

$$\bar{T}_{\mu\nu}(\bar{x}) = \bar{S}_{\mu\nu}(t, \theta, \varphi)\delta(\bar{\eta}) = \bar{S}_{\mu\nu}(t, \theta, \varphi)\sqrt{1 - \frac{2m}{r_1}}\,\delta(\eta), \tag{48}$$

where the last equation in (47) has been used. Therefore, we find for the pressure-defining parts of the "thin shell" stress–energy tensor ($\bar{S}_{ij} = \bar{\mathcal{P}}G_{ij}$ and $S_{ij} = \mathcal{P}G_{ij}$, respectively) following the coordinate transformation relation:

$$\bar{S}_{ij} = \left(1 - \frac{2m}{r_1}\right)^{-\frac{1}{2}}S_{ij} \tag{49}$$

with S_{ij} – the same as in (43). Thus, for the pressure $\bar{\mathcal{P}}$ as defined within the standard approach using Gaussian normal coordinate $\bar{\eta}$ we have:

$$\bar{\mathcal{P}} = \frac{1}{4\pi r_1}\frac{1 - m/r_1}{\sqrt{1 - 2m/r_1}} \longrightarrow \infty \quad \text{for } r_1 \to 2m, \tag{50}$$

which exactly corresponds to Eq. (15.46) in Ref. [2] and diverges in the lightlike limit due to the prefactor on the r.h.s. of (49), whereas for the pressure $\mathcal{P}$ as defined within our choice of coordinates we get:

$$\mathcal{P} = \bar{\mathcal{P}}\sqrt{1 - 2m/r_1} = \frac{1}{4\pi r_1}\left(1 - \frac{m}{r_1}\right)$$

$$\longrightarrow \frac{1}{16\pi m} = \text{finite} \quad \text{for } r_1 \to 2m, \tag{51}$$

which precisely agrees with the second equation in (44) and with (37)–(38). Accordingly, the "thin-shell" energy density:

$$\rho = \bar{\rho}\sqrt{1 - 2m/r_1} = -\frac{1}{2\pi r_1}\left(1 - \frac{2m}{r_1}\right) \tag{52}$$

is negative and vanishes in the lightlike limit ($r_1 \to 2m$) in agreement with (38) (the expression for $\bar{\rho}$ in (52) is the same as in the standard treatment using the metric (45), *cf.* Eq. (15.45) in Ref. [2]).

The inappropriateness of the Gaussian normal coordinate $\bar{\eta}$ in the lightlike limit can also be seen by observing that in the vicinity of the horizon it coincides with the original Einstein–Rosen coordinate u in (1).

6. Conclusions

The original Einstein–Rosen "bridge" manifold [1], namely, two identical copies of the outer Schwarzschild space–time region glued together along the Schwarzschild horizon, appears as a particular case of the general construction of spherically and axially symmetric traversable wormholes produced by *LL-branes* as gravitational sources proposed in Refs. [4,5]. Here "traversability" means that a travelling observer crosses the wormhole throat from the one "universe" to the other one within a finite amount of his/her *proper* time. The same traversability property exists also w.r.t. the Eddington–Finkelstein time v (*cf.* (5)).

The main lesson, as explained in some detail above, is that consistency of Einstein equations of motion yielding the original Einstein–Rosen "bridge" as well-defined solution necessarily requires the presence of *LL-brane* energy–momentum tensor as a source on the right-hand side. Thus, the introduction of *LL-brane* coupling to gravity brings the original Einstein–Rosen construction in Ref. [1] to a consistent completion.

Codimension one *LL-branes* possess natural couplings to bulk Maxwell $\mathcal{A}_\mu$ and Kalb–Ramond $\mathcal{A}_{\mu_1 \ldots \mu_{p+1}}$ gauge fields ($D-1 = p+1$, see Ref. [9]):

$$
\widetilde{S}_{LL} = \int d^{p+1}\sigma \, \Phi(\varphi) \left[-\frac{1}{2} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + L(F^2) \right]
$$
$$
- q \int d^{p+1}\sigma \, \varepsilon^{ab_1 \ldots b_p} F_{b_1 \ldots b_p} \partial_a X^\mu \mathcal{A}_\mu(X)
$$
$$
- \frac{\beta}{(p+1)!} \int d^{p+1}\sigma \, \varepsilon^{a_1 \ldots a_{p+1}} \partial_{a_1} X^{\mu_1} \cdots
$$
$$
\times \, \partial_{a_{p+1}} X^{\mu_{p+1}} \mathcal{A}_{\mu_1 \ldots \mu_{p+1}}(X), \tag{53}
$$

where q indicates the surface charge density of the *LL-brane*. As shown in [9], the *LL-brane* can serve as a material and charge source for gravity and electromagnetism by coupling it to bulk Einstein–Maxwell + Kalb–Ramond-field system:

$$
S = \int d^D x \sqrt{-G} \left[\frac{R(G)}{16\pi} - \frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right.
$$
$$
\left. - \frac{1}{D!2} \mathcal{F}_{\mu_1 \ldots \mu_D} \mathcal{F}^{\mu_1 \ldots \mu_D} \right] + \widetilde{S}_{LL} \tag{54}
$$

with $\widetilde{S}_{LL}$ given by (53). Moreover, the *LL-brane* generates *dynamical* cosmological constant through the coupling to the Kalb–Ramond bulk field: $\Lambda = 4\pi\beta^2$. One can employ the above formalism to construct more general asymmetric traversable wormholes, *e.g.*, glueing together an exterior Schwarzschild region ("left" universe) with an exterior Reissner–Nordström region ("right" universe) where the throat is the standard Schwarzschild horizon w.r.t.

the "left" universe and it is simultaneously the *outer* Reissner–Nordström horizon w.r.t. the "right" universe. Again as above "traversability" means traversability w.r.t. the proper time of travelling observers crossing the throat.

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